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ON THE PROOFS OF THE FUNDAMENTAL THEOREM ON ALMOST PERIODIC FUNCTIONS

BY

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1. Introduction. The fundamental theorem of the theory of almost periodic functions states that any almost periodic function $f(x)$ with the Fourier series

$$f(x) \sim \sum a(\lambda)\, e^{i\lambda x}, \text{ where } a(\lambda) = M\{f(x)\, e^{-i\lambda x}\},$$

satisfies the Parseval equation

$$M\{|f(x)|^2\} = \Sigma\, |a(\lambda)|^2.$$

Many proofs of this theorem have been given. Among them the proof of Weyl [6] is, perhaps, the one which leads most directly to the goal. It depends on a systematic use of the process of convolution and on the methods of the theory of integral equations. Another proof, depending on the general theory of Fourier integrals, is due to Wiener [7]; it has been given a particularly simple form by Bochner [2, pp. 81—82].

Though these proofs give a clear insight in the whole theory, the more elementary proofs are not without interest. Among them the original proof of Bohr [3] is interesting by its crudeness. Its idea is to consider for every positive T the periodic function with the period T which coincides with $f(x)$ in the interval $(0, T)$, and to use Parseval's equation for this function. By making $T \to \infty$, one obtains the theorem. The passage to the limit is, however, of a complicated nature, and the whole proof is very long.

A considerable simplification was obtained by de la Vallée Poussin [5], who used the same idea together with the convolution process to prove the uniqueness theorem, which states that if $a(\lambda) = 0$ for all λ, then $f(x)$ vanishes identically. From this theorem Parseval's equation follows by a simple application of the convolution process. Since $f(x)$ vanishes identically if and only if $M\{|f(x)|^2\} = 0$, the proof of the uniqueness theorem amounts to a proof of Parseval's equation in the particular case where $a(\lambda) = 0$ for all λ. A simplification of de la Vallée Poussins proof has been given by M. Riesz [4].

It seems very natural to base a proof of the fundamental theorem on almost periodic functions on the corresponding theorem on periodic functions. It must, however, be mentioned, that the periodic function with the period T which coincides with $f(x)$ in the interval $(0, T)$ will generally be discontinuous in the points nT, so that it is not a special case of the theorem on almost periodic functions, which is used. Moreover, the periodic functions will generally not approximate $f(x)$ outside the interval $(0, T)$.

The truth is that actually it complicates matters to introduce this periodic function. As will be shown in the following pages, the proofs take a simpler form if, instead of the Fourier series of the periodic function with the period T, we consider the Fourier integral of the function $f_T(x)$ which coincides with $f(x)$ in the interval $(0, T)$ and is 0 elsewhere. Naturally, for large T this Fourier integral does not differ much from the Fourier series of the periodic function.

All we shall need on Fourier integrals is, that if $F(x)$ is a function, which is continuous in a certain closed interval and is 0 outside this interval, and if

$$\int_{-\infty}^{\infty} F(x)\, e^{-i\lambda x}\, dx = A(\lambda),$$

then in analogy to Parseval's equation

$$\int_{-\infty}^{\infty} |F(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |A(\lambda)|^2\, d\lambda .$$

Thus our proofs are more elementary than the proof of Wiener referred to above, with which they have no connection.

Our proof of the Parseval equation follows step by step Bohr's proof. The main simplifications are in the beginning. In the later part a simplification in the exposition has been obtained by the use of a function introduced by Wiener [7, p. 495], connected with Bochner's translation function [1, p. 136].

In our proof of the uniqueness theorem we use de la Vallée Poussin's main lemma, which actually concerns the Fourier integral of the function $f_T(x)$. The simplification is in the remainder of the proof, where we avoid the artifice of choosing T as a fine translation number.

2. Proof of the Parseval equation. The inequality obtained from Parseval's equation by replacing $=$ by $\geq$ being an easy consequence of Bessel's formula, it is sufficient to prove the inequality obtained by replacing $=$ by $\leq$.

For an arbitrary $T > 0$ consider the function

$$\frac{1}{T}\int_0^T f(x)\, e^{-i\lambda x}\, dx = \frac{1}{T}\int_{-\infty}^{\infty} f_T(x)\, e^{-i\lambda x}\, dx = a_T(\lambda).$$

Then

$$\frac{1}{T}\int_0^T |f(x)|^2\, dx = \frac{1}{T}\int_{-\infty}^{\infty} |f_T(x)|^2\, dx = \frac{T}{2\pi}\int_{-\infty}^{\infty} |a_T(\lambda)|^2\, d\lambda.$$

It is therefore sufficient to prove:

To every $\delta > 0$ there exists a $T_0 > 0$ such that

$$\frac{T}{2\pi}\int_{-\infty}^{\infty} |a_T(\lambda)|^2\, d\lambda < \Sigma |a(\lambda)|^2 + \delta \quad \textit{for} \quad T > T_0.$$

3. We begin by proving

Lemma 1. *To every λ_0 and every $\delta > 0$ correspond an $\omega > 0$ and a $T_0 > 0$ such that*

$$\frac{T}{2\pi}\int_{\lambda_0-\omega}^{\lambda_0+\omega} |a_T(\lambda)|^2\, d\lambda < |a(\lambda_0)|^2 + \delta \quad \textit{for} \quad T > T_0.$$

Proof. If $f(x)$ is replaced by $f(x)\, e^{-i\lambda_0 x}$ the function $a_T(\lambda)$ is replaced by $a_T(\lambda + \lambda_0)$. It is therefore sufficient to consider the case $\lambda_0 = 0$.

(i) $a(0) = 0$, i. e. $M\{f(x)\} = 0$. — On placing for a given $c > 0$

$$\Phi(x) = \frac{1}{c}\int_x^{x+c} f_T(y)\,dy$$

we have by a simple computation

$$\frac{1}{T}\int_{-\infty}^{\infty} \Phi(x)\, e^{-i\lambda x}\,dx = a_T(\lambda)\,\frac{e^{i\lambda c}-1}{i\lambda c};$$

hence

$$\frac{1}{T}\int_{-\infty}^{\infty} |\Phi(x)|^2\,dx = \frac{T}{2\pi}\int_{-\infty}^{\infty} |a_T(\lambda)|^2 \left|\frac{e^{i\lambda c}-1}{i\lambda c}\right|^2 d\lambda.$$

Since $\frac{1}{c}\int_x^{x+c} f(y)\,dy$ converges uniformly in x towards $M\{f(x)\}$ as $c \to \infty$ there exists to every $\varepsilon > 0$ a $c = c(\varepsilon)$, such that when $T > c$ then $|\Phi(x)| \leq \varepsilon$ in the interval $(0,\, T-c)$. In the intervals $(-c,\, 0)$ and $(T-c,\, T)$ we have $|\Phi(x)| \leq G = \sup|f(x)|$. Outside the interval $(-c,\, T)$ we have $\Phi(x) = 0$. Hence

$$\frac{1}{T}\int_{-\infty}^{\infty} |\Phi(x)|^2\,dx \leq \varepsilon^2 + \frac{2\,c\,G^2}{T}$$

and consequently

$$\frac{T}{2\pi}\int_{-\infty}^{\infty} |a_T(\lambda)|^2 \left|\frac{e^{i\lambda c}-1}{i\lambda c}\right|^2 d\lambda \leq 2\,\varepsilon^2 \quad \text{for} \quad T > T_0 = \frac{2\,c\,G^2}{\varepsilon^2}.$$

For $|\lambda| <$ (some) $\omega = \omega(c)$ we have

$$\left|\frac{e^{i\lambda c}-1}{i\lambda c}\right| > \frac{1}{2}.$$

Hence

$$\frac{T}{2\pi}\int_{-\omega}^{\omega} |a_T(\lambda)|^2\,d\lambda \leq 4\cdot 2\,\varepsilon^2 = 8\,\varepsilon^2 \quad \text{for} \quad T > T_0.$$

(ii) $a(0) = a \neq 0$. — On placing $f(x) = a + h(x)$ we obtain a corresponding decomposition of $a_T(x)$ in two terms:

$$a_T(\lambda) = b_T(\lambda) + c_T(\lambda).$$

Hereby

$$\frac{1}{T}\int_0^T |a|^2\,dx = |a|^2 = \frac{T}{2\pi}\int_{-\infty}^{\infty} |b_T(\lambda)|^2\,d\lambda.$$

Hence by the triangle inequality we have for every $\omega > 0$

$$\left[\frac{T}{2\pi}\int_{-\omega}^{\omega} |a_T(\lambda)|^2\,d\lambda\right]^{\frac{1}{2}} \leqq |a| + \left[\frac{T}{2\pi}\int_{-\omega}^{\omega} |c_T(\lambda)|^2\,d\lambda\right]^{\frac{1}{2}}.$$

Let $\varepsilon > 0$ be chosen such that $(|a| + 3\varepsilon)^2 < |a|^2 + \delta$, and next by (i), since $M\{h(x)\} = 0$, the numbers ω and T_0 such that

$$\frac{T}{2\pi}\int_{-\omega}^{\omega} |c_T(\lambda)|^2\,d\lambda < 9\,\varepsilon^2 \quad \text{for} \quad T > T_0.$$

Then

$$\frac{T}{2\pi}\int_{-\omega}^{\omega} |a_T(\lambda)|^2\,d\lambda \leqq |a|^2 + \delta \quad \text{for} \quad T > T_0.$$

4. On account of Lemma 1 it is, in order to prove the Parseval equation, sufficient to prove the following

Lemma 2. *To every* $\delta > 0$ *there exists a finite set of numbers* $\lambda_1, \cdots, \lambda_M$ *such that for every* $\omega > 0$

$$\frac{T}{2\pi}\int_{\substack{|\lambda-\lambda_1| \geqq \omega \\ \cdots \\ |\lambda-\lambda_M| \geqq \omega}} |a_T(\lambda)|^2\,d\lambda < \delta \quad \textit{for} \quad T > (\textit{some})\; T_0(\omega).$$

We shall reduce this lemma to a lemma on the translation function

$$e(\tau) = \sup_x |f(x+\tau) - f(x)|.$$

On placing for a given $\tau > 0$

$$\psi(x) = f_T(x+\tau) - f_T(x)$$

we have

$$\frac{1}{T}\int_{-\infty}^{\infty} \psi(x)\,e^{-i\lambda x}\,dx = a_T(\lambda)\,(e^{i\lambda\tau} - 1);$$

hence
$$\frac{1}{T}\int_{-\infty}^{\infty}|\Psi(x)|^2\,dx = \frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\,|e^{i\lambda\tau}-1|^2\,d\lambda.$$

Now $|\Psi(x)| \leq e(\tau)$ if the points x and $x+\tau$ both lie in the interval $(0, T)$, and $|\Psi(x)| \leq G$ if one of the points lie in $(0, T)$, whereas $\Psi(x) = 0$ if both points lie outside $(0, T)$. Hence
$$\frac{1}{T}\int_{-\infty}^{\infty}|\Psi(x)|^2\,dx \leq e(\tau)^2 + \frac{2\tau G^2}{T}$$

and consequently
$$\frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\,|e^{i\lambda\tau}-1|^2\,d\lambda \leq 2\varepsilon^2 \quad \text{if} \quad e(\tau) \leq \varepsilon \quad \text{and} \quad T > \frac{2\tau G^2}{\varepsilon^2}.$$

On placing
$$\varphi(\tau) = \begin{cases} \varepsilon - e(\tau) & \text{when } e(\tau) \leq \varepsilon \\ 0 & \text{when } e(\tau) > \varepsilon \end{cases}$$

we therefore obtain for every $X > 0$
$$\frac{T}{2\pi}\int_{-\infty}^{\infty}|a_T(\lambda)|^2\left(\frac{1}{X}\int_0^X |e^{i\lambda\tau}-1|^2\,\varphi(\tau)\,d\tau\right)d\lambda < 2\varepsilon^2\,\frac{1}{X}\int_0^X \varphi(\tau)\,d\tau$$
$$\text{for} \quad T > \frac{2XG^2}{\varepsilon^2}.$$

In order to prove Lemma 2 it is therefore sufficient to prove

Lemma 3. *To every $\varepsilon > 0$ there exists a finite set of numbers $\lambda_1, \cdots, \lambda_M$ such that for every $\omega > 0$ there exists an $X > 0$ for which*
$$\frac{1}{X}\int_0^X |e^{i\lambda\tau}-1|^2\,\varphi(\tau)\,d\tau > \frac{1}{17}\,\frac{1}{X}\int_0^X \varphi(\tau)\,d\tau$$

when $|\lambda-\lambda_1| \geq \omega, \cdots, |\lambda-\lambda_M| \geq \omega$.

5. Translation numbers of $f(x)$ belonging to a given $\varrho > 0$, i. e. numbers τ for which $e(\tau) \leq \varrho$, will be denoted throughout by $\tau(\varrho)$. We shall first prove

Lemma 4. *For every* $\varrho > 0$ *the set* S *of numbers* λ *for which* $|e^{i\lambda t} - 1| \leqq 1$ *for all* $t = \tau(\varrho) > 0$ *is finite.*

If S *consist of the numbers* $\lambda_1, \cdots, \lambda_M$, *there exists to every* $\omega > 0$ *an* $A > 0$, *such that if* $|\lambda - \lambda_1| \geqq \omega, \cdots, |\lambda - \lambda_M| \geqq \omega$, *then* $|e^{i\lambda t} - 1| > 1$ *for some positive* $t = \tau(\varrho) < A$.

Proof. By the uniform continuity of $f(x)$ there exists an $\eta > 0$ such that any positive $\tau < \eta$ is a $\tau(\varrho)$. Hence, if $|\lambda| > \pi/3\,\eta$ there exists a positive $t = \tau(\varrho) < \eta$ for which $|e^{i\lambda t} - 1| > 1$. Thus S belongs to the interval $|\lambda| \leqq \pi/3\,\eta$.

If λ' and λ'' both belong to S, i. e. if $|\lambda' t| \leqq \pi/3 \mod 2\pi$ and $|\lambda'' t| \leqq \pi/3 \mod 2\pi$ for all $t = \tau(\varrho) > 0$, we have $|\lambda' - \lambda''|\, t \leqq 2\pi/3 \mod 2\pi$ for all $t = \tau(\varrho) > 0$. In particular, the interval $2\pi/3|\lambda' - \lambda''| < t < 4\pi/3|\lambda' - \lambda''|$ of length $2\pi/3|\lambda' - \lambda''|$ will contain no $\tau(\varrho)$. Since every interval of a certain length $l = l(\varrho)$ contains a $\tau(\varrho)$, we obtain $|\lambda' - \lambda''| \geqq 2\pi/3\,l$. Hence S is finite.

Let now $\omega > 0$ be chosen, and consider the closed bounded set of numbers λ for which $|\lambda - \lambda_1| \geqq \omega, \cdots, |\lambda - \lambda_M| \geqq \omega$, $|\lambda| \leqq \pi/3\,\eta$. This set is covered by the open sets U_t defined by an inequality $|e^{i\lambda t} - 1| > 1$ for a $t = \tau(\varrho) > 0$. Hence, by Borel's theorem, it is covered by a finite number of these sets, say by $U_{t_1}, \cdots, U_{t_n}$. As number A may then be used any number larger than the numbers $\eta, t_1, \cdots, t_n$.

6. We now turn to the proof of Lemma 3.

The translation function $e(\tau)$ being almost periodic, so is the function $\varphi(\tau)$. Since $\varphi(\tau)$ is non-negative and not identically zero, we have

$$M\{\varphi(\tau)\} = m > 0.$$

In Lemma 4 let $\varrho = \frac{1}{2} m$. Then the lemma gives numbers $\lambda_1, \cdots, \lambda_M$ and, when $\omega > 0$ is chosen, a number $A > 0$.

If $|\lambda - \lambda_1| \geqq \omega, \cdots, |\lambda - \lambda_M| \geqq \omega$, there exists a positive $t = \tau(\varrho) < A$ such that $|e^{i\lambda t} - 1| > 1$. For $X > A$ we have

$$\frac{1}{X}\int_0^X |e^{i\lambda\tau} - 1|^2 \varphi(\tau)\, d\tau \geqq \frac{1}{2X}\int_0^{X-A} \left[|e^{i\lambda\tau} - 1|^2 \varphi(\tau) + |e^{i\lambda(\tau+t)} - 1|^2 \varphi(\tau + t)\right] d\tau.$$

Now, since $|\lambda t| > \pi/3 \mod 2\pi$, the relations $|\lambda\tau| \leqq \pi/6 \mod 2\pi$ and $|\lambda(\tau + t)| \leqq \pi/6 \mod 2\pi$ cannot be valid together, i. e. we have for every τ

$$\max\{|e^{i\lambda\tau}-1|^2, |e^{i\lambda(\tau+t)}-1|^2\} > |e^{i\pi/6}-1|^2 > \tfrac{1}{4}.$$

Moreover, since t is a $\tau(\varrho)$, we have $e(\tau+t) \leq e(\tau)+\varrho$, and consequently

$$\varphi(\tau+t) \geq \varphi(\tau)-\varrho.$$

Hence we obtain

$$\frac{1}{X}\int_0^X |e^{i\lambda\tau}-1|^2 \varphi(\tau)\, d\tau \geq \frac{1}{2X}\int_0^{X-A} \tfrac{1}{4}[\varphi(\tau)-\varrho]\, d\tau.$$

Here the right hand side converges for $X\to\infty$ towards $\frac{1}{8}(m-\varrho) = \frac{1}{16}m$. The right hand side of the inequality in Lemma 3 converges for $X\to\infty$ towards $\frac{1}{17}m$. Hence the latter inequality is valid for some X and the proof is completed.

7. Proof of the uniqueness theorem. The main lemma in de la Vallée Poussin's proof states that when $a(\lambda)=0$ for all λ, then $a_T(\lambda)\to 0$ uniformly in λ as $T\to\infty$. Starting from this lemma the proof may by completed as follows.

For a given $\varepsilon>0$ let $T_0>0$ be chosen such that $|a_T(\lambda)| \leq \varepsilon$ for all λ when $T>T_0$. For $U>T>T_0$ consider the function

$$g_{TU}(x) = \frac{1}{T}\int_{-\infty}^{\infty} f_U(x+t)\overline{f_T(t)}\, dt = \frac{1}{T}\int_0^T f_U(x+t)\overline{f(t)}\, dt.$$

Plainly, $g_{TU}(x)$ vanishes outside the interval $(-T, U)$ and coincides in the interval $(0, U-T)$ with the almost periodic function

$$g_T(x) = \frac{1}{T}\int_0^T f(x+t)\overline{f(t)}\, dt.$$

By a simple calculation

$$\int_{-\infty}^{\infty} g_{TU}(x) e^{-i\lambda x}\, dx = U a_U(\lambda)\overline{a_T(\lambda)}.$$

Hence

$$\frac{1}{U}\int_0^{U-T} |g_T(x)|^2\, dx \leq \frac{1}{U}\int_{-\infty}^{\infty} |g_{TU}(x)|^2\, dx = \frac{U}{2\pi}\int_{-\infty}^{\infty} |a_U(\lambda)|^2 |a_T(\lambda)|^2\, d\lambda$$

$$\leq \varepsilon^2 \frac{U}{2\pi}\int_{-\infty}^{\infty} |a_U(\lambda)|^2\, d\lambda = \varepsilon^2 \frac{1}{U}\int_0^U |f(x)|^2\, dx \leq \varepsilon^2 G^2.$$

For $U \to \infty$ this gives

$$M\{|g_T(x)|^2\} \leqq \varepsilon^2 G^2.$$

For $T \to \infty$ the function $g_T(x)$ converges uniformly in x towards the convolution

$$g(x) = \underset{t}{M}\{f(x+t)\overline{f(t)}\}.$$

Hence

$$M\{|g(x)|^2\} \leqq \varepsilon^2 G^2.$$

Since this is true for all $\varepsilon > 0$, we have $M\{|g(x)|^2\} = 0$, which implies $g(x) \equiv 0$. In particular $g(0) = M\{|f(x)|^2\} = 0$, and hence $f(x) \equiv 0$.

8. Another variant of the proof of the uniqueness theorem. It may be remarked that a slight change in the above proof permits us to replace the use of Parseval's formula for Fourier integrals by Parseval's formula for periodic functions, which may be formulated as follows:

If $F(x)$ is continuous in a closed interval of length $\leqq P$ and is 0 outside this interval, and if

$$\int_{-\infty}^{\infty} F(x)\, e^{-i\lambda x}\, dx = A(\lambda),$$

then

$$\int_{-\infty}^{\infty} |F(x)|^2\, dx = \frac{1}{P} \sum_{n=-\infty}^{\infty} \left| A\left(\frac{2\pi}{P} n\right)\right|^2.$$

Applying this formula to the function $g_{TU}(x)$, which vanishes outside the interval $(-T, U)$, and using that $f_U(x)$ also vanishes outside this interval we obtain

$$\int_0^{U-T} |g_T(x)|^2\, dx \leqq \int_{-\infty}^{\infty} |g_{TU}(x)|^2\, dx = \frac{1}{T+U} \sum_{n=-\infty}^{\infty} U^2 \left| a_U\left(\frac{2\pi}{T+U} n\right)\right|^2 \left| a_T\left(\frac{2\pi}{T+U} n\right)\right|^2$$

$$\leqq \varepsilon^2 \frac{1}{T+U} \sum_{n=-\infty}^{\infty} U^2 \left| a_U\left(\frac{2\pi}{T+U} n\right)\right|^2 = \varepsilon^2 \int_{-\infty}^{\infty} |f_U(x)|^2\, dx \leqq \varepsilon^2\, U G^2,$$

and the proof is completed as before.

References.

[1] S. Bochner. Beiträge zur Theorie der fastperiodischen Funktionen I. Math. Ann. 96 (1926), pp. 119—147.

[2] — Vorlesungen über Fouriersche Integrale. Leipzig 1932.

[3] H. Bohr. Zur Theorie der fastperiodischen Funktionen I. Acta math. 45 (1924), pp. 29—127.

[4] M. Riesz. Eine Bemerkung über den Eindeutigkeitssatz der Theorie der fastperiodischen Funktionen. Mat. Tidsskrift B 1934, pp. 11—13.

[5] C. de la Vallée Poussin. Sur les fonctions presque périodiques de H. Bohr. Ann. Soc. Sci. Bruxelles 47_2 (1927), pp. 141—158; 48_1 (1928), pp. 56—57.

[6] H. Weyl. Integralgleichungen und fastperiodische Funktionen. Math. Ann. 97 (1926), pp. 338—356.

[7] N. Wiener. The spectrum of an arbitrary function. Proc. London Math. Soc. (2) 27 (1928), pp. 483—496.

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ON THE UNIQUENESS OF THE POTENTIAL IN A SCHRÖDINGER EQUATION FOR A GIVEN ASYMPTOTIC PHASE

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1. The solution $y(x, \lambda)$ of the differential equation

$$y'' + (\lambda^2 - P(x))\, y = 0 \tag{1.0}$$

with initial values

$$y(0, \lambda) = 0, \quad y'(0, \lambda) = 1, \tag{1.1}$$

will, in case $P(x)$ is small enough for large x, and in case $\lambda = u \neq 0$ where u is real, satisfy

$$\lim_{x \to \infty} \left[y(x, u) - \frac{A(u)}{u} \sin(ux - \Phi(u)) \right] = 0 \tag{1.2}$$

where $A(u)$ and $\Phi(u)$ are continuous function of u, $0 < u < \infty$. The function $\Phi(u)$ is the asymptotic phase.

The problem of determining the potential $P(x)$ from $\Phi(u)$ arises in physics. Recently C. E. Fröberg, [1], has given various approximate procedures for calculating $P(x)$ from $\Phi(u)$ based on the variation of constants formula or on one or more iterations of this formula. He treats the equation $y'' - \frac{l(l+1)}{x^2} y + (\lambda^2 - (Px))\, y = 0$ where l is an integer. The equation (1.0) is the case $l = 0$. Fröberg observes that his method need not of course be convergent. Indeed the question arises as to whether $\Phi(u)$ determines $P(x)$ uniquely at all. We shall show that with suitable hypotheses this is indeed the case. We shall also see that $\Phi(u)$ determines $A(u)$ in (1.2) uniquely and conversely. The theory we shall develop for (1.0) can be carried over to more general cases. (See note on p. 27 for the case $l > 0$.)

Theorem I. *If $P(x)$ is piecewise continuous (or more generally if $P(x)$ is Lebesgue measurable), if*

(1.3) $$P(x) \geqq 0$$

and

(1.4) $$\int_0^\infty x \, | P(x) | \, dx < \infty ,$$

then (1.2) *is valid where* $A(u)$ *and* $\Phi(u)$ *are continuous functions of* u. *There is no other potential function* $Q(x)$ *satisfying the same hypothesis as* $P(x)$ *with an identical phase function* $\Phi(u)$. *Moreover* $\Phi(u)$ *determines* $A(u)$ *uniquely and conversely.*

The condition (1.3) can be modified. However, without (1.3) it is possible for (1.0) to have discrete characteristic values $\lambda_k = iv_k$, $k = 1, 2, \cdots$, where the v_k are real. Associated with each $\lambda_k = iv_k$ there is exactly one characteristic function $y(x, \lambda_k)$ which for large x is $O(e^{-v_k x})$. If we assume

(1.5) $$\int_1^\infty x^2 \, | P(x) | \, dx < \infty$$

in place of (1.3) then we shall find that there are at most a finite number of characteristic values, $\lambda_k = iv_k$, and with $v_k \neq 0$.

We shall see that under the hypothesis of Theorem II, if

$$\Phi(\infty) - \Phi(+0) < \pi,$$

then there are no discrete characteristic values. (In fact we shall find that we always have either $\Phi(\infty) = \Phi(+0) + m\pi$ or $\Phi(\infty) = \Phi(+0) + \left(m + \frac{1}{2}\right)\pi$ where m is the number of characteristic values in $v > 0$, i. e. with $\lambda^2 < 0$). We now have the following result.

Theorem II. *If* $P(x)$ *is real and measureable and if*

(1.6) $$\int_0^1 x \, | P(x) | \, dx + \int_1^\infty x^2 \, | P(x) | \, dx < \infty$$

then (1.2) *is valid. If there are no discrete characteristic values, i. e., if* $\Phi(\infty) - \Phi(+0) < \pi$, *then there is no potential function* $Q(x)$ *different from* $P(x)$ *satisfying* (1.6) *and with the same phase function* $\Phi(u)$. *Moreover* $\Phi(u)$ *determines* $A(u)$ *uniquely and conversely.*

In case $P(x)$ satisfies

$$\int_0^1 |P(x)|\, dx < \infty \tag{1.7}$$

which is a stricter requirement at $x = 0$ than (1.4) or (1.6) it is possible to consider initial values of the form

$$y(0, \lambda) = \sin \alpha, \quad y'(0, \lambda) = \cos \alpha. \tag{1.8}$$

In this case we could dispense with some of the lemmas we require for Theorems I and II and use known results [2, § 5.3 and Chapter VI] in their place. The methods used here will carry over to cover (1.8) with the assumption (1.7). However, in practise the condition

$$\int_0^1 x\, |P(x)|\, dx < \infty$$

is much more useful than (1.7) and we shall carry out our proofs for this case.

We shall see in the course of our proof that the spectral representation of a function $f(x)$ involves the integral

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u^2}{A^2(u)}\, y(x, u)\, du \int_0^{\infty} y(\xi, u)\, f(\xi)\, d\xi.$$

Thus we see that the weight function in this integral $u^2/A^2(u)$ determines $A(u)$, and therefore from theorems I and II also $\Phi(u)$. Thus the weight function $u^2/A^2(u)$ can arise from one $P(x)$ only.

In the course of our proof we shall also get the following relationship valid for any function $f(x)$ in $L^2(0, \infty)$,

$$\int_0^{\infty} |f(x)|^2\, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u^2}{A^2(u)}\, du \left| \int_0^{\infty} y(x, u)\, f(x)\, dx \right|^2. \tag{1.9}$$

We shall see that there is a function of $\lambda = u + iv$, $F(\lambda)$, analytic for $v > 0$ and continuous for $v \geqq 0$ such that for real

$\lambda = u$ we have $F(u) = A(u)\, e^{i\Phi(u)}$. We shall see that the behavior of $F(\lambda)$ as $\lambda \to 0$ is of concern to us and for this reason we need requirement (1.3) in Theorem I and

$$\int_1^\infty x^2 \,|\, P(x) \,|\, dx < \infty$$

in Theorem II.

2. Here we shall show that $\Phi(u)$ determines $A(u)$ and conversely under the hypothesis of Theorem I. Actually we shall use only (1.4) except to show that $F(0) \neq 0$ where we need (1.3). Thus most of § 2 will be available to us to prove Theorem II as well.

We shall require the following results. We shall use K to denote positive constants which depend on $P(x)$ only. We recall $\lambda = u + iv$.

Lemma 2.0. *If $P(x)$ satisfies (1.4) then there is a solution $y(x, \lambda)$ of (1.0) satisfying (1.1) which for any x is an entire function of λ and which for all λ satisfies*

$$|\, y(x, \lambda) \,| \leqq \frac{Kxe^{|v|x}}{1 + |\lambda|\, x}, \quad 0 \leqq x < \infty. \tag{2.0}$$

As $|\lambda| \to \infty$

$$y(x, \lambda) = \frac{\sin \lambda x}{\lambda} + o\left(\frac{e^{|v|x}}{|\lambda|}\right) \tag{2.1}$$

uniformly in x, $0 \leqq x < \infty$. Moreover $y(x, \lambda)$ is an even function of λ.

Lemma 2.1. *If $P(x)$ satisfies (1.4) then for $v \geqq 0$ there is a solution of (1.0), $y_1(x, \lambda)$ which for each x is an analytic function of λ for $v > 0$ and continuous for $v \geqq 0$ and satisfies*

$$|\, y_1(x, \lambda) \,| \leqq Ke^{-vx}, \quad 0 \leqq x < \infty \tag{2.2}$$

and

$$|\, y_1(x, \lambda) - e^{i\lambda x} \,| \leqq \frac{Ke^{-vx}}{|\lambda|} \int_x^\infty |\, P(\xi) \,|\, d\xi. \tag{2.3}$$

For $v \leqq 0$ *there exists a function* $y_2(x, \lambda)$ *similarly related to* $e^{-i\lambda x}$ *for large* x *or* $|\lambda|$.

We shall prove these lemmas in § 5. It is clear that for $\lambda = u$, $y_1(x, u)$ and $y_2(x, u)$ by (2.3) are independent solutions of (1.0) for large x. Since they are independent for large x, they are independent for all x, $0 \leqq x < \infty$. From (2.0) we have

$$(2.4) \qquad |y(x, \lambda)| \leqq Kxe^{|v|x}.$$

We also have, as can be verified by substitution into (1.0), the "variation of constants" formula

$$(2.5) \qquad y(x, \lambda) = \frac{\sin \lambda x}{\lambda} + \frac{1}{\lambda}\int_0^x \sin \lambda (x-\xi) P(\xi) y(\xi, \lambda) d\xi.$$

Here the right side exists because of (2.4). We see from (2.5), by use of (2.4), that for $\lambda = u \neq 0$ we have as $x \to \infty$

$$(2.6) \qquad \begin{cases} y(x, u) = \dfrac{\sin ux}{u}\left(1 + \displaystyle\int_0^\infty \cos u\xi P(\xi) y(\xi, u) d\xi\right) \\ \qquad\qquad + \dfrac{\cos ux}{u}\displaystyle\int_0^\infty \sin u\xi P(\xi) y(\xi, u) d\xi + o(1). \end{cases}$$

Or as $x \to \infty$

$$y(x, u) = \frac{A(u)}{u}\sin(ux - \Phi(u)) + o(1)$$

where if

$$F(u) = 1 + \int_0^\infty e^{iu\xi} P(\xi) y(\xi, u) d\xi$$

then by (2.6)

$$A(u) = |F(u)|, \quad \Phi(u) = \arg F(u).$$

Since $y(x, u)$ is an even function of u, $A(u)$ is an even function. From (1.4) and (2.4) we see that

$$(2.7) \qquad F(\lambda) = 1 + \int_0^\infty e^{i\lambda\xi} P(\xi) y(\xi, \lambda) d\xi$$

is an analytic function of λ for $v > 0$ and is continuous for $v \geqq 0$. The properties of $F(\lambda)$ are given in the following lemma.

Lemma 2.2. *If $P(x)$ is real and satisfies* (1.4) *then* $F(\lambda)$ *defined in* (2.7) *is analytic in the half-plane* $v > 0$ *and continuous for* $v \geqq 0$. *In* $v \geqq 0$ *it can vanish only for values of* λ *for which* $u = 0$. *If* $\lambda_k = iv_k$, $v_k > 0$, *is a root of* $F(\lambda) = 0$ *then* $y(x, iv_k)$ *is a characteristic solution of* (1.0) *satisfying, for some* $C_k \neq 0$,

$$(2.8) \qquad y(x, iv_k) = C_k y_1(x, iv_k) \to 0 \quad as \quad x \to \infty .$$

For large $|\lambda|$ *we have*

$$(2.9) \qquad F(\lambda) = 1 + o(1)$$

uniformly for $0 \leqq \arg \lambda \leqq \pi$.

The proof of lemma 2.2 is given in § 5.

For $v > 0$ we have the following relationship for $e^{i\lambda x} y(x, \lambda)$ as $x \to \infty$. From (2.5) we have

$$\left\{ \begin{aligned} & e^{i\lambda x} y(x, \lambda) = e^{i\lambda x} \frac{\sin \lambda x}{\lambda} \\ & + \frac{1}{\lambda} \int_0^x \sin \lambda (x - \xi)\, e^{i\lambda (x-\xi)} P(\xi)\, y(\xi, \lambda)\, e^{i\lambda \xi} d\xi. \end{aligned} \right.$$

Letting $x \to \infty$ and using (2.4) we get

$$(2.10) \qquad \lim_{x \to \infty} e^{i\lambda x} y(x, \lambda) = -\frac{F(\lambda)}{2i\lambda}.$$

We shall now introduce the hypothesis $P(x) \geqq 0$ and show that in this case $F(iv) \neq 0$ for $v \geqq 0$. We have

$$(2.11) \qquad F(iv) = 1 + \int_0^\infty e^{-v\xi} P(\xi)\, y(\xi, iv)\, d\xi.$$

Since $y'' = (v_1^2 + P) y$, $y(0, iv_1) = 0$ and $y'(0, iv_1) = 1$ we see that $y'' \geqq 0$ and thus $y' \geqq 1$ and $y \geqq 0$. In (2.11) this yields $F(iv) \geqq 1$.

Since $F(\lambda) \neq 0$ for $v \geqq 0$ and since $F(\lambda) = 1 + o(1)$ as $|\lambda| \to \infty$ uniformly for $0 \leqq \arg \lambda \leqq \pi$, wee see that $g(\lambda) = \log F(\lambda)$ is analytic for $v > 0$ and continuous for $v \geqq 0$ and moreover we can choose $g(\lambda)$ so that

$$g(\lambda) = o(1) \quad \text{as} \quad |\lambda| \to \infty \tag{2.12}$$

uniformly for $0 \leq \arg \lambda \leq \pi$. Applying Cauchy's theorem over a semi-circle of radius R with center at $\lambda = 0$ and diameter on the real axis and letting $R \to \infty$ we find by use of (2.12) that

$$g(\lambda) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{g(\sigma)}{\sigma - \lambda} d\sigma \tag{2.13}$$

where σ is real and $\lambda = u + iv$, $v > 0$. In the same way if $\bar{\lambda} = u - iv$, $v > 0$ then

$$0 = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{g(\sigma)}{\sigma - \bar{\lambda}} d\sigma. \tag{2.14}$$

Taking the conjugate of the latter formula and adding to (2.13) we find

$$g(\lambda) = \lim_{R \to \infty} \frac{1}{\pi} \int_{-R}^{R} \frac{\operatorname{Im} g(\sigma)}{\sigma - \lambda} d\sigma.$$

Or since $\operatorname{Im} g(\sigma) = \Phi(\sigma)$

$$\log F(\lambda) = \lim_{R \to \infty} \frac{1}{\pi} \int_{-R}^{R} \frac{\Phi(\sigma)}{\sigma - \lambda} d\sigma. \tag{2.15}$$

Thus we see that $\Phi(u)$ determines $F(\lambda)$ and in particular then, $\Phi(u)$ determines

$$A(u) = \lim_{v \to +0} |F(u + iv)|.$$

We observe that $\Phi(u)$ is an odd function of u. By subtracting the conjugate of (2.14) from (2.13) and using the fact that $A(u)$ is even we get for $v > 0$

$$\log F(\lambda) = \frac{2\lambda}{\pi i} \int_{0}^{\infty} \frac{\log A(\sigma)}{\sigma^2 - \lambda^2} d\sigma.$$

Thus $A(u)$ determines $F(\lambda)$ and in particular then determines

$$\Phi(u) = \lim_{v \to +0} \operatorname{Im} \log F(u + iv).$$

3. We now assume that there is another differential equation with $P(x)$ replaced by $Q(x)$ where $Q(x)$ satisfies the same hypothesis as $P(x)$ in Theorem I and where the asymptotic phase is again $\Phi(u)$. The equation is

$$z'' + (\lambda^2 - Q(x))z = 0. \tag{3.0}$$

Since the asymptotic phase of $z(x, u)$ is $\Phi(u)$, its asymptotic amplitude is $A(u)/u$. Thus

$$z(x, u) = \frac{A(u)}{u} \sin(ux - \Phi(u)) + o(1) \tag{3.1}$$

as $x \to \infty$ where $z(0, u) = 0$ and $z'(0, u) = 1$. There are also two solutions of (3.0), $z_1(x, \lambda)$ and $z_2(x, \lambda)$ satisfying the same conditions as y_1 and y_2 in Lemma 2.1.

Returning to $y(x, u)$ where $u \neq 0$ we have since y_1 and y_2 are independent solutions.

$$y(x, u) = C_1(u)\, y_1(x, u) + C_2(u)\, y_2(x, u).$$

Letting $x \to \infty$ we have

$$\frac{A(u)}{u} \sin(ux - \Phi(u)) = C_1(u)\, e^{iux} + C_2(u)\, e^{-iux} + o(1).$$

From this we see that indeed the term $o(1)$ is zero and that

$$C_1(u) = \frac{A(u)\, e^{-i\Phi(u)}}{2iu}, \quad C_2(u) = -\frac{A(u)\, e^{i\Phi(u)}}{2iu}.$$

Thus

$$y(x, u) = \frac{A(u)}{2iu}\left[y_1(x, u)\, e^{-i\Phi(u)} - y_2(x, u)\, e^{i\Phi(u)}\right]. \tag{3.2}$$

In exactly the same way we see that (3.1) implies

$$z(x, u) = \frac{A(u)}{2iu}\left[z_1(x, u)\, e^{-i\Phi(u)} - z_2(x, u)\, e^{i\Phi(u)}\right]. \tag{3.3}$$

Let $f(x)$ be a real differentiable function which vanishes for

x near zero and for large x. Let $\max(|f(x)|+|f'(x)|) = M$. (These requirements on $f(x)$ are somewhat more severe than is actually necessary for our argument.) We now consider the following pseudo-Green's function integrals of $f(x)$,

$$\text{(3.4)} \quad \begin{cases} H_1(x,\lambda) = \dfrac{\lambda}{F(\lambda)} y(x,\lambda) \displaystyle\int_x^\infty z_1(\xi,\lambda) f(\xi)\, d\xi \\ H_2(x,\lambda) = \dfrac{\lambda}{F(\lambda)} z_1(x,\lambda) \displaystyle\int_0^x y(\xi,\lambda) f(\xi)\, d\xi. \end{cases}$$

Clearly for each $x, H_j, j = 1, 2$, is analytic in λ in the upper half-plane $v > 0$ and continuous for $v \geqq 0$. Thus if c is the semi-circle of radius R, $\lambda = Re^{i\theta}$, $0 \leqq \theta \leqq \pi$, then for any x, $0 < x < \infty$, Cauchy's theorem yields

$$\text{(3.5)} \qquad \int_c H_j(x,\lambda)\, d\lambda + \int_{-R}^{R} H_j(x,u)\, du = 0.$$

Let $\delta > 0$ and let

$$J_1 = \int_x^\infty z_1(\xi,\lambda) f(\xi)\, d\xi = \left(\int_x^{x+\delta} + \int_{x+\delta}^\infty\right) z_1(\xi,\lambda) f(\xi)\, d\xi.$$

Using (2.3) we get since $\int_x^\infty |P(\xi)|\, d\xi \leqq \int_x^\infty \xi\, |P(\xi)|\, d\xi / x$,

$$\left| J_1 - \int_x^{x+\delta} e^{i\lambda\xi} f(\xi)\, d\xi \right| \leqq \frac{Ke^{-vx}}{|\lambda| x} \int_x^{x+\delta} |f(\xi)|\, d\xi + \frac{KMe^{-(x+\delta)v}}{|\lambda|}\left(1 + \frac{1}{x}\right).$$

Integrating by parts we have

$$\left| \int_x^{x+\delta} e^{i\lambda\xi} f(\xi)\, d\xi + \frac{e^{i\lambda x} f(x)}{i\lambda} \right| \leqq \frac{M\delta e^{-vx}}{|\lambda|} + \frac{Me^{-(x+\delta)v}}{|\lambda|}.$$

Thus

$$\text{(3.6)} \qquad \left| J_1 + \frac{e^{i\lambda x} f(x)}{i\lambda} \right| \leqq \frac{KMe^{-vx}}{|\lambda|} \left(\delta + e^{-\delta v}\right)\left(1 + \frac{1}{x}\right).$$

Therefore for large $|\lambda| = R$, using (2.1) (2.9) and (3.6)

$$\left| \int_c \frac{\lambda\, y(x,\lambda)}{F(\lambda)}\, d\lambda \int_x^\infty z_1(\xi,\lambda) f(\xi)\, d\xi + \frac{f(x)}{i} \int_c \frac{y(x,\lambda)\, e^{i\lambda x}}{F(\lambda)}\, d\lambda \right|$$
$$\leq KM\left(\delta + \frac{1}{R\delta}\right)\left(1 + \frac{1}{x}\right).$$

If $\delta = R^{-1/2}$ we have uniformly in x for any closed interval of x interior to the open interval $(0, \infty)$,

$$\lim_{R\to\infty} \int_c \frac{\lambda\, y(x,\lambda)}{F(\lambda)}\, d\lambda \int_x^\infty z_1(\xi,\lambda) f(\xi)\, d\xi = -\frac{1}{2}\pi i f(x).$$

Using this in (3.5) with $j = 1$ we have

$$f(x) = \frac{2}{\pi i} \lim_{R\to\infty} \int_{-R}^{R} \frac{u\, y(x,u)}{F(u)}\, du \int_x^\infty z_1(\xi,u) f(\xi)\, d\xi. \tag{3.7}$$

We also have the following result.

Lemma 3.0. *For any* $x > 0$,

$$\frac{\lambda z_1(x,\lambda)}{F(\lambda)} \int_0^x y(\xi,\lambda) f(\xi)\, d\xi = -\frac{f(x)}{2\lambda} + J_2(x,\lambda)$$

where uniformly for any closed interval of x interior to the open interval $(0, \infty)$

$$\lim_{R\to\infty} \int_c |J_2(x,\lambda)|\, |d\lambda| = 0.$$

The proof of this lemma is given in § 5.

Using Lemma 3.0 we have

$$\lim_{R\to\infty} \int_c \frac{\lambda z_1(x,\lambda)}{F(\lambda)}\, d\lambda \int_0^x y(\xi,\lambda) f(\xi)\, d\xi = -\frac{1}{2}\pi i f(x). \tag{3.8}$$

Using (3.8) in (3.5) with $j = 2$ we have

$$f(x) = \frac{2}{\pi i} \lim_{R\to\infty} \int_{-R}^{R} \frac{u z_1(x,u)}{F(u)}\, du \int_0^x y(\xi,u) f(\xi)\, d\xi. \tag{3.9}$$

Since $\bar{z}_1(x, u)$ is a solution of (3.0), $\bar{z}_1(x, u) = C_1 z_1(x, u) + C_2 z_2(x, u)$. Letting $x \to \infty$ we see that $\bar{z}_1(x, u) = z_2(x, u)$. Taking the conjugate of (3.7) and (3.9) we have therefore

$$(3.10)\quad f(x) = -\frac{2}{\pi i} \lim_{R\to\infty} \int_{-R}^{R} \frac{u y(x, u)}{\overline{F}(u)} du \int_x^\infty z_2(\xi, u) f(\xi)\, d\xi,$$

$$(3.11)\quad f(x) = -\frac{2}{\pi i} \lim_{R\to\infty} \int_{-R}^{R} \frac{u z_2(x, u)}{\overline{F}(u)} du \int_0^x y(\xi, u) f(\xi)\, d\xi.$$

Since by (3.3)

$$z(x, u) = \frac{A^2(u)}{2iu}\left(\frac{z_1(x, u}{F(u)} - \frac{z_2(x, u)}{\overline{F}(u)}\right)$$

we have by adding (3.7) and (3.10)

$$(3.12)\quad f(x) = \lim_{R\to\infty} \frac{2}{\pi} \int_{-R}^{R} \frac{u^2 y(x, u)}{A^2(u)} du \int_x^\infty z(\xi, u) f(\xi)\, d\xi.$$

In the same way (3.9) and (3.11) give

$$(3.13)\quad f(x) = \lim_{R\to\infty} \frac{2}{\pi} \int_{-R}^{R} \frac{u^2 z(x, u)}{A^2(u)} du \int_0^x y(\xi, u) f(\xi)\, d\xi.$$

Interchanging the role of y and z we get instead of (3.12)

$$f(x) = \lim_{R\to\infty} \frac{2}{\pi} \int_{-R}^{R} \frac{u^2 z(x, u)}{A^2(u)} du \int_x^\infty y(\xi, u) f(\xi)\, d\xi.$$

Combining the above with (3.13) we have

$$(3.14)\quad f(x) = \lim_{R\to\infty} \frac{1}{\pi} \int_{-R}^{R} \frac{u^2 du}{A^2(u)} z(x, u) \int_0^\infty y(\xi, u) f(\xi)\, d\xi.$$

Since the convergence above is uniform except near $x = 0$ and $x = \infty$ where $f(x)$ vanishes we have

$$(3.15)\quad \int_0^\infty f^2(x)\, dx = \lim_{R\to\infty} \frac{1}{\pi} \int_{-R}^{R} \frac{u^2 du}{A^2(u)} \int_0^\infty z(x, u) f(x)\, dx \int_0^\infty y(\xi, u) f(\xi)\, d\xi.$$

The derivation of (3.15) is certainly valid if z is replaced by y. Thus

$$\int_0^\infty f^2(x)\,dx = \lim_{R\to\infty}\frac{1}{\pi}\int_{-R}^{R}\frac{u^2du}{A^2(u)}\left(\int_0^\infty y(x,u)\,f(x)\,dx\right)^2 \tag{3.16}$$

and the corresponding result with y replaced by z. Combining we get

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u^2}{A^2(u)}\,du\left(\int_0^\infty y(x,u)\,f(x)\,dx - \int_0^\infty z(x,u)\,f(x)\,dx\right)^2 = 0.$$

Thus

$$\int_0^\infty y(x,u)\,f(x)\,dx = \int_0^\infty z(x,u)\,f(x)\,dx. \tag{3.17}$$

For any fixed u let us suppose $z(x,u) \neq y(x,u)$ at $x = x_1 > 0$. Let us suppose then that $y(x_1,u) - z(x_1,u) > 0$ for some u. Since $y(x,u)$ and $z(x,u)$ are differentiable they are continuous and we must have for small $\delta > 0$, where $x_1 - \delta > 0$,

$$y(x,u) - z(x,u) > 0, \quad |x - x_1| \leqq \delta.$$

Choose $f(x) > 0$ for $|x - x_1| < \delta$ and $f(x) = 0$ for $|x - x_1| \geqq \delta$. Then clearly for the value of u in question

$$\int_0^\infty [y(x,u) - z(x,u)]\,f(x)\,dx > 0$$

which contradicts (3.17). The same argument applies of course if $z - y > 0$ and we see then that $y(x,u) = z(x,u)$. Therefore from the differential equations for y and z we get

$$y(x,u)\,[P(x) - Q(x)] = 0. \tag{3.18}$$

(In case $P(x)$ or $Q(x)$ are discontinuous (3.18) holds almost everywhere.) Since $y(x,u)$ vanishes only for isolated values of x and since xP and xQ are integrable we have $P(x) = Q(x)$ almost everywhere which proves Theorem I.

4. Here we no longer assume $P(x) \geqq 0$ but rather that $\int_1^\infty x^2\,|P(x)|\,dx < \infty$ and proceed to prove Theorem II. Lemma 2.2

is valid but the argument in § 2 which follows it can of course no longer be used. Now we shall show that either $F(0) \neq 0$ or else that near $\lambda = 0$ and for $v \geqq 0$

$$F(\lambda) = \alpha\lambda + o(|\lambda|) \text{ where } \alpha \neq 0. \tag{4.0}$$

Since $F(\lambda) \to 1$ for large $|\lambda|$ and is analytic for $v > 0$ and since the zeros of $F(\lambda)$ all lie on the line $u = 0$ we see that *either* $F(0) \neq 0$ *or* (4.0) *implies that there are at most a finite number of zeros of* $F(\lambda)$ *in the upper half-plane.*

With $F(0) = 0$ we also have

$$F(\lambda) = \int_0^\infty e^{i\lambda x} y(\lambda, x) P(x)\, dx - \int_0^\infty y(x, 0) P(x)\, dx. \tag{4.1}$$

Thus

$$\left\{\begin{aligned} F(\lambda) &= \int_0^\infty (e^{i\lambda x} - 1)\, y(x, 0) P(x)\, dx \\ &\quad + \int_0^\infty e^{i\lambda x} [y(x, \lambda) - y(x, 0)] P(x)\, dx = I_1 + I_2. \end{aligned}\right. \tag{4.2}$$

Here

$$I_1 = \int_0^\infty (e^{i\lambda x} - 1)\, y(x, 0) P(x)\, dx \tag{4.3}$$

and I_2 represents the second integral in (4.2). We have from (1.0) when $F(0) = 0$

$$y'(x, 0) = 1 + \int_0^x P(\xi) y(\xi, 0)\, d\xi = -\int_x^\infty P(\xi) y(\xi, 0)\, d\xi. \tag{4.4}$$

Or since by Lemma 2.2, $|y(x, 0)| \leqq Kx$,

$$|y'(x, 0)| \leqq K \int_x^\infty \xi |P(\xi)|\, d\xi.$$

Thus

$$\int_1^\infty |y'(x, 0)|\, dx \leqq K \int_1^\infty dx \int_x^\infty \xi |P(\xi)|\, d\xi = K \int_1^\infty \xi^2 |P(\xi)|\, d\xi < \infty.$$

From this we see that

$$|y(x, 0)| < K. \tag{4.5}$$

From (4.3)

$$\left| I_1 - i\lambda \int_0^\infty x\, y(x, 0)\, P(x)\, dx \right|$$

$$\leq \int_0^\infty | e^{i\lambda x} - 1 - i\lambda x | \, | y(x, 0)\, P(x) | \, dx$$

$$\leq |\lambda|^2 \int_0^{1/|\lambda|} x^2 | y(x, 0)\, P(x) | \, dx + 3|\lambda| \int_{1/|\lambda|}^\infty x | y(x, 0)\, P(x) | \, du,$$

wherein the last integral above we use $| e^{i\lambda x} - 1 | \leq 2$ for $v \geq 0$ and $2 \leq 2|\lambda| x$ for $x \geq 1/|\lambda|$. Using (4.5) we see that as $|\lambda| \to 0$

(4.6) $$I_1 - i\lambda \int_0^\infty xy(x, 0)\, P(x)\, dx = o(|\lambda|).$$

Now we shall show

(4.7) $$\int_0^\infty xy(x, 0)\, P(x)\, dx \neq 0.$$

We have from (4.4)

(4.8) $$y(x, 0) = x + \int_0^x (x - \xi)\, P(\xi)\, y(\xi, 0)\, d\xi.$$

If (4.7) is false and if $F(0) = 0$ then (4.8) becomes

(4.9) $$y(x, 0) = -x \int_x^\infty P(\xi)\, y(\xi, 0)\, d\xi + \int_x^\infty \xi P(\xi)\, y(\xi, 0)\, d\xi.$$

Let $x_1 > 1$ be large enough so

$$\int_{x_1}^\infty \xi | P(\xi) | \, d\xi < \frac{1}{4}.$$

Let $\max\limits_{x \geq x_1} | y(x, 0) | = m$. Then by (4.9)

$$m \leq 2 \int_{x_1}^\infty \xi | P(\xi)\, y(\xi, 0) | \, d\xi \leq \frac{1}{2} m.$$

Thus $m = 0$ which is impossible and we see then that (4.7) holds. Thus from (4.6) we have as $\lambda \to 0$

(4.10) $$I_1 = \alpha\lambda + o(|\lambda|) \text{ where } \alpha \neq 0.$$

We show next that $I_2 = o(|\lambda|)$ as $\lambda \to 0$. We have

(4.11) $$I_2 = \int_0^\infty e^{i\lambda x} [y(x, \lambda) - y(x, 0)]\, P(x)\, dx.$$

As solutions of

$$y'' - P(x)\,y = 0 \tag{4.12}$$

we have $y_3(x) = y(x, 0)$ and an independent solution $y_4(x)$ chosen so that $y_3 y_4' - y_4 y_3' = 1$. Since by (4.4), $y_3(x) - x \to 0$ as $x \to 0$ we see that a solution of (4.12) independent of y_3 is

$$y(x) = y_3(x) \int_x \frac{d\xi}{y_3^2(\xi)}$$

and this is bounded as $x \to 0$. Thus y_4 is bounded as $x \to 0$. We have obviously also

$$y_4(x) = y_4(x_1) + (x - x_1)\,y_4'(x_1) + \int_{x_1}^{x} (x - \xi)\,P(\xi)\,y_4(\xi)\,d\xi.$$

If $\max\limits_{x_1 \leq x \leq x_2} \left| \dfrac{y_4(x)}{x} \right| = m$ and if x_1 is chosen as below (4.9) then clearly for large x_2

$$m \leqq |y_4(x_1)| + |y_4'(x_1)| + m \int_{x_1}^{\infty} \xi\,|P(\xi)|\,d\xi.$$

Thus

$$\frac{3}{4}m \leqq |y_4'(x_1)| + |y_4(x_1)|$$

and we see that $|y_4(x)| \leqq Kx$ for large x. Now if

$$y'' - P(x)\,y = f(x)$$

then

$$y(x) = c_1 y_3(x) + c_2 y_4(x) - \int_0^x [y_3(x)\,y_4(\xi) - y_4(x)\,y_3(\xi)]\,f(\xi)\,d\xi.$$

Thus from

$$y''(x, \lambda) - P(x)\,y(x, \lambda) = -\lambda^2 y(x, \lambda)$$

we have

$$y(x, \lambda) = y_3(x) + \lambda^2 \int_0^x [y_3(x)\,y_4(\xi) - y_4(x)\,y_3(\xi)]\,y(\xi, \lambda)\,d\xi.$$

Thus

$$I_2 = \lambda^2 \int_0^{\infty} e^{i\lambda x} P(x)\,dx \int_0^x [y_3(x)\,y_4(\xi) - y_4(x)\,y_3(\xi)]\,y(\xi, \lambda)\,d\xi.$$

Or

$$|I_2| \leqq K|\lambda|^2 \int_0^\infty x|P(x)|\,dx \int_0^x |y(\xi,\lambda)|\,e^{-\nu x}\,d\xi .$$

Using Lemma 2.0 we have

$$|I_2| \leqq K|\lambda|^2 \int_0^\infty x|P(x)|\,dx \int_0^x \frac{\xi}{1+|\lambda|\,\xi}\,d\xi .$$

Thus

$$|I_2| \leqq K|\lambda|^2 \int_0^{1/|\lambda|} x|P(x)|\,dx \int_0^x \xi\,d\xi$$

$$+K|\lambda|^2 \int_{1/|\lambda|}^\infty x\,|\,P(x)\,|\,dx \left(\int_0^{1/|\lambda|} \xi\,d\xi + \int_{1/|\lambda|}^x \frac{d\xi}{|\lambda|}\right)$$

$$\leqq K|\lambda|^2 \int_0^{1/|\lambda|} x^3|P(x)|\,dx + K\int_{1/|\lambda|}^\infty x|P(x)|\,dx + K|\lambda| \int_{1/|\lambda|}^\infty x^2|P(x)|\,dx$$

$$\leqq K|\lambda|^{3/2} \int_0^{1/|\lambda|^{1/2}} x^2|P(x)|\,dx + K|\lambda| \int_{1/|\lambda|^{1/2}}^{1/|\lambda|} x^2|P(x)|\,dx$$

$$+2K|\lambda| \int_{1/|\lambda|}^\infty x^2|P(x)|\,dx .$$

Since $\int_0^\infty x^2|P(x)|\,dx < \infty$ we have then as $|\lambda| \to 0$

$$I_2 = o(|\lambda|).$$

Thus we have demonstrated (4.0).

Exactly as in § 2 we find that if $F(0) \neq 0$ then the formula

$$\log F(\lambda) = \lim_{R\to\infty} \frac{1}{\pi} \int_R^{-R} \frac{\Phi(\sigma)}{\sigma-\lambda}\,d\sigma$$

is valid as are the other formulae. In this case we have, since $F(\lambda) \to 1$, as $|\lambda| \to \infty$, that the total number of zeros of $F(\lambda)$

in $v > 0$ is given by $(\Phi(\infty) - \Phi(-\infty))/2\pi = (\Phi(\infty) - \Phi(0))/\pi$. Since $\Phi(+0) = \Phi(0)$ here we see that if $\Phi(\infty) - \Phi(+0) < \pi$ the total number of zeros must be zero and in fact that $\Phi(\infty) = \Phi(0)$. Since $\Phi(\infty)$ can be taken as zero we see that $\Phi(0) = 0$ and thus if $F(0) \neq 0$, $F(0) > 0$. If $F(0) = 0$, then we can work with a contour containing a small semi-circle, γ, with center at $\lambda = 0$ and radius ϱ. On γ, $\lambda = \varrho e^{i\theta}$, $0 \leqq \theta \leqq \pi$. Thus as in (2.13) $g(\lambda) = \log F(\lambda)$ is given by

$$g(\lambda) = \lim_{R \to \infty} \frac{1}{2\pi i} \left(\int_{-R}^{-\varrho} + \int_{\varrho}^{R} \right) \frac{g(\sigma)}{\sigma - \lambda} d\sigma + \frac{1}{2\pi i} \int_{\gamma} \frac{g(\sigma)}{\sigma - \lambda} d\sigma.$$

Since $g(\lambda) = \log \frac{F(\lambda)}{\lambda} + \log \lambda$ near $\lambda = 0$ we find on letting $\varrho \to 0$ that since $\frac{F(\lambda)}{\lambda} \to \alpha \neq 0$ we have

$$g(\lambda) = \lim_{R \to \infty} \frac{1}{2\pi i} \int_{-R}^{R} \frac{g(\sigma)}{\sigma - \lambda} d\sigma$$

from which all the other formulas relating F, A and Φ follow. Here we find that $\Phi(+0) - \Phi(-0) = -\pi$ and that the total number of zeros m of $F(\lambda)$ in $v > 0$ is given by

$$m = \frac{1}{2\pi}(\Phi(\infty) - \Phi(+0) + \Phi(-0) - \Phi(-\infty)) - \frac{1}{2}.$$

Since $\Phi(\infty) - \Phi(+0) = \Phi(-0) - \Phi(-\infty)$ we have $m = \frac{1}{\pi}(\Phi(\infty) - \Phi(+0)) - \frac{1}{2}$. Since $\Phi(\infty) - \Phi(+0) < \pi$ we see that $m = 0$. In fact here we must have $\Phi(\infty) = \Phi(+0) + \frac{1}{2}\pi$. Since we take $\Phi(\infty) = 0$ we have $\Phi(+0) = -\frac{1}{2}\pi$. Also $\Phi(-0) = \frac{1}{2}\pi$ and as in the other case $\Phi(u)$ is an odd function. The results of § 3 carry over without change thus establishing Theorem II.

5. Here we prove a number of lemmas. In all these lemmas we shall require only that

$$(5.0) \qquad \int_0^\infty x\,|P(x)|\,dx < \infty .$$

In the proofs of Lemmas 2.0 and 2.1 the formulas are written for the case $\lambda \neq 0$. In case $\lambda = 0$ the changes involved are obvious.

Proof of Lemma 2.0. Consider the sequence $y_n(x, \lambda)$ where $y_0(x, \lambda) = 0$ and

$$(5.1) \; y_n(x, \lambda) = \frac{\sin \lambda x}{\lambda} + \frac{1}{\lambda}\int_0^x \sin \lambda (x-\xi)\, P(\xi)\, y_{n-1}(\xi, \lambda)\, d\xi .$$

We have if $\lambda = u + iv$, for $v \geqq 0$

$$(5.2) \qquad |y_1(x, \lambda) - y_0(x, \lambda)| = \left|\frac{\sin \lambda x}{\lambda}\right| = xe^{vx}\left|\frac{1-e^{2i\lambda x}}{\lambda x}\right| .$$

Thus

$$(5.3) \qquad |y_1 - y_0| = \left|\frac{\sin \lambda x}{\lambda}\right| \leqq 4\, xe^{|v|x}$$

and this is true for all λ. Using this in (5.1) we get

$$|y_2 - y_1| \leqq \int_0^x \left|\frac{\sin \lambda (x-\xi)}{\lambda}\right| |P(\xi)|\, 4\,\xi e^{|v|\xi}\, d\xi .$$

Much as we found (5.3) we have

$$(5.4) \qquad \left|\frac{\sin \lambda (x-\xi)}{\lambda}\right| \leqq 4\, xe^{|v|(x-\xi)}, \quad 0 \leqq \xi \leqq x .$$

Thus

$$|y_2 - y_1| \leqq 4^2 xe^{|v|x}\int_0^x \xi\,|P(\xi)|\,d\xi .$$

If we set

$$B(x) = \int_0^x \xi\,|P(\xi)|\,d\xi < \int_0^\infty x\,|P(x)|\,dx$$

then

$$|y_2 - y_1| \leqq 4^2 xe^{|v|x} B(x) .$$

Again from (5.1) we have

$$|y_3 - y_2| \leqq 4^3 xe^{|v|x}\int_0^x \xi\,|P(\xi)|\,B(\xi)\,d\xi = 4^3 xe^{|v|x}\frac{(B(x))^2}{2!} .$$

Proceeding we have

$$|y_n - y_{n-1}| \leqq 4^n x e^{|v|x} \frac{(B(x))^{n-1}}{(n-1)!}.$$

Thus in any finite range of x and finite region of λ, $y_n(x, \lambda)$ converges uniformly to a limit $y(x, \lambda)$ which is therefore analytic in λ and by (5.1) satisfies

$$(5.5) \quad y(x, \lambda) = \frac{\sin \lambda x}{\lambda} + \frac{1}{\lambda} \int_0^x \sin \lambda (x - \xi) P(\xi) y(\xi, \lambda) d\xi.$$

From (5.5) we see easily that $y(x, \lambda)$ is a solution of (1.0) and satisfies

$$y(0, \lambda) = 0, \quad y'(0, \lambda) = 1.$$

Let

$$(5.6) \qquad M(x, \lambda) = \frac{1}{x} |y(x, \lambda)| (1 + |\lambda| x) e^{-|v|x}.$$

From (5.2) we have easily by considering separately $|\lambda x| \leqq 1$ and $|\lambda x| > 1$,

$$\left|\frac{\sin \lambda x}{\lambda}\right| \leqq \frac{8 x e^{|v|x}}{1 + |\lambda| x}.$$

Using this in (5.5) we have

$$\frac{x M(x, \lambda) e^{|v|x}}{1 + |\lambda| x} \leqq \frac{8 x e^{|v|x}}{1 + |\lambda| x} + 8 \int_0^x \frac{(x - \xi) e^{|v|(x-\xi)}}{1 + |\lambda|(x - \xi)} |P(\xi)| \frac{M(\xi, \lambda) \xi e^{|v|\xi}}{1 + |\lambda| \xi} d\xi.$$

Or

$$M(x, \lambda) \leqq 8 + 8 \int_0^x \xi |P(\xi)| M(\xi, \lambda) d\xi.$$

By a well known inequality this implies

$$M(x, \lambda) \leqq 8 \exp\left(8 \int_0^x \xi |P(\xi)| d\xi\right).$$

Using (5.0) we have $M(x, \lambda) \leqq K$. With (5.6) this gives

$$(5.7) \qquad |y(x, \lambda)| \leqq \frac{K x e^{|v|x}}{1 + |\lambda| x}.$$

In (5.5) this gives

$$\left\{\begin{aligned} \left| y(x,\lambda) - \frac{\sin\lambda x}{\lambda} \right| &\leqq \frac{K e^{|v|x}}{|\lambda|} \int_0^x \frac{|\xi|}{1+|\lambda|\xi} |P(\xi)|\, d\xi \\ &\leqq \frac{K e^{|v|x}}{|\lambda|} \left(\int_0^{1/|\lambda|^{1/2}} \xi |P(\xi)|\, d\xi + \frac{1}{|\lambda|^{1/2}} \int_{1/|\lambda|^{1/2}}^{\infty} \xi |P(\xi)|\, d\xi \right). \end{aligned}\right.$$

Thus as $|\lambda| \to \infty$

$$y(x,\lambda) = \frac{\sin\lambda x}{\lambda} + o\left(\frac{e^{|v|x}}{|\lambda|}\right).$$

That $y(x,\lambda)$ is an even function of λ is clear from the fact that each $y_n(x,\lambda)$ is even. The completes the proof of the lemma.

Proof of Lemma 2.1. Let $W_0(x,\lambda) = 0$ and let

$$\text{(5.8)} \quad W_n(x,\lambda) = e^{i\lambda x} - \frac{1}{\lambda} \int_x^{\infty} \sin\lambda(x-\xi)\, P(\xi)\, W_{n-1}(\xi,\lambda)\, d\xi.$$

From (5.3) for $v \geqq 0$

$$\left| \frac{\sin\lambda(x-\xi)}{\lambda} \right| \leqq 4(\xi - x)\, e^{v(\xi - x)}, \quad \xi \geqq x.$$

Clearly $|W_1 - W_0| \leqq e^{-vx}$ and

$$\begin{aligned} |W_2(x,\lambda) - W_1(x,y)| &\leqq 4 \int_x^{\infty} (\xi - x)\, e^{v(\xi - x)} |P(\xi)|\, e^{-v\xi}\, d\xi \\ &\leqq 4\, e^{-vx} \int_x^{\infty} \xi |P(\xi)|\, d\xi. \end{aligned}$$

If

$$B(x) = \int_x^{\infty} \xi |P(\xi)\, d\xi$$

then

$$|W_2 - W_1| \leqq 4\, e^{-vx} B(x).$$

Again

$$\left\{\begin{aligned} &|W_3 - W_2|\, 4^2 e^{-vx} \int_x^{\infty} \xi |P(\xi)|\, B(\xi)\, d\xi \\ &= 4^2 e^{-vx} \frac{(B(x))^2}{2!} \leqq 4^2 e^{-vx} \frac{(B(o))^2}{2!} \end{aligned}\right.$$

etc. Thus $W_n(x,\lambda)$ converges uniformly for $v \geqq 0$ and $0 \leqq x < \infty$ to a limit we denote by $y_1(x,\lambda)$. Clearly

$$|y_1(x,\lambda)| \leq Ke^{-vx}$$

and from (5.8)

$$(5.9)\qquad y_1(x,\lambda) = e^{i\lambda x} - \frac{1}{\lambda}\int_x^\infty \sin\lambda(x-\xi)\,P(\xi)\,y_1(\xi,\lambda)\,d\xi.$$

From this we have

$$(5.10)\qquad |y_1(x,\lambda) - e^{i\lambda x}| \leq \frac{Ke^{-vx}}{|\lambda|}\int_x^\infty |P(\xi)|\,d\xi.$$

This proves the lemma.

Proof of Lemma 2.2. That

$$F(\lambda) = 1 + \int_0^\infty e^{i\lambda x}\,y(x,\lambda)\,P(x)\,dx$$

is analytic for $v > 0$ and continuous for $v \geq 0$ follows from Lemma 2.0 and (5.0). That $F(\lambda) = 1 + o(1)$ uniformly in $v \geq 0$ as $|\lambda| \to \infty$ follows from use of

$$\left\{\begin{aligned} |F(\lambda) - 1| &\leq K\int_0^\infty \frac{\xi}{1+|\lambda|\,\xi}\,|P(\xi)|\,d\xi \\ &\leq K\int_0^{1/|\lambda|^{1/2}} \xi\,|P(\xi)|\,d\xi + \frac{K}{|\lambda|^{1/2}}\int_{1/|\lambda|^{1/2}}^\infty \xi\,|P(\xi)|\,d\xi. \end{aligned}\right.$$

For real $\lambda = u \neq 0$ we have as in (3.2)

$$y(x,u) = \frac{A(u)}{2iu}\left[y_1(x,u)\,e^{-i\Phi(u)} - y_2(x,u)\,e^{i\Phi u}\right].$$

If $F(u) = 0$ then $A(u) = 0$ and $y(x,u) = 0$ which is impossible. Thus $F(u) \neq 0$ for $u \neq 0$.

Let $F(\lambda)$ vanish for some $\lambda_1 = u_1 + iv_1$, $v_1 > 0$. For large x

$$y_3(x,\lambda_1) = -2i\lambda_1\,y_1(x,\lambda_1)\int^x \frac{d\xi}{y_1^2(\xi,\lambda_1)}$$

is a solution of (1.0). Since $y_1(x,\lambda_1) \sim e^{i\lambda_1 x}$ we have

$$y_3(x,\lambda_1) \sim e^{-i\lambda_1 x}$$

as $x \to \infty$. Moreover from (5.9) we also get

$$y_1'(x, \lambda) \sim i\lambda_1 e^{i\lambda_1 x}. \tag{5.11}$$

Since y_1 and y_3 are obviously independent

$$y(x, \lambda_1) = c_1 y_1(x, \lambda_1) + c_2 y_3(x, \lambda_1).$$

If $F(\lambda_1) = 0$ we see from (2.10) that we must have $c_2 = 0$. Thus

$$y(x, \lambda_1) = c_1 y_1(x, \lambda_1) \sim c_1 e^{i\lambda_1 x} \tag{5.12}$$

and from (5.11)

$$y'(x, \lambda_1) \sim ic_1 \lambda_1 e^{i\lambda_1 x}. \tag{5.13}$$

Using a familiar argument we have that the conjugate of $y(x, \lambda_1)$, $\bar{y}(x, \lambda_1)$ is a solution of (1.0) with λ_1 replaced by $\bar{\lambda}_1 = u_1 - iv_1$. Thus

$$\left\{ \begin{aligned} & y(x, \lambda_1)\,\bar{y}'(x, \lambda_1) - \bar{y}(x, \lambda_1)\,y'(x, \lambda_1) \\ & \quad + (\bar{\lambda}_1^2 - \lambda_1^2) \int_0^x |y(x, \lambda_1)|^2\,dx = 0. \end{aligned} \right.$$

Letting $x \to \infty$ and using (5.12) and (5.13) we have

$$(\bar{\lambda}_1^2 - \lambda_1^2) \int_0^\infty |y(x, \lambda_1)|^2\,dx = 0.$$

Thus $u_1 = 0$ and $\lambda_1 = iv_1$ if $F(\lambda_1) = 0$. This completes the proof of the lemma.

We prove finally

Lemma 3.0. We have by (5.5)

$$\left\{ \begin{aligned} J = {} & \frac{\lambda z_1(x, \lambda)}{F(\lambda)} \int_0^x y(\xi, \lambda) f(\xi)\,d\xi = \frac{z_1(x, \lambda)}{F(\lambda)} \int_0^x \sin\lambda\xi\, f(\xi)\,d\xi \\ & + \frac{z_1(x, \lambda)}{F(\lambda)} \int_0^x f(\xi)\,d\xi \int_0^\xi \sin\lambda(\xi - s)\,P(s)\,y(s, \lambda)\,ds = I_1 + I_2. \end{aligned} \right. \tag{5.14}$$

Clearly on integrating by parts

$$I_1 = \frac{z_1(x, \lambda)}{F(\lambda)} \left[-f(x) \frac{\cos\lambda x}{\lambda} + \frac{1}{\lambda} \int_0^x \cos\lambda\xi\, f'(\xi)\,d\xi \right].$$

Thus for large $|\lambda|$ and $v \geqq 0$ using (2.2) and (2.9)

$$\left| I_1 + \frac{z_1(x,\lambda) f(x) \cos \lambda x}{\lambda F(\lambda)} \right| \leqq \frac{K e^{-\delta v}}{|\lambda|} \int_0^{x-\delta} |f'(\xi)\, d\xi| + \frac{KM\delta}{|\lambda|}$$

where we recall $M = \max\, (|f(x)| + |f'(x)|)$. Or by (5.10) and the above inequality for large $|\lambda|$

$$(5.15)\quad \left\{ \begin{aligned} \left| I_1 + \frac{1}{2\lambda} f(x) \right| &\leqq \left| I_1 + \frac{1}{2\lambda} \frac{1}{F(\lambda)} f(x) \right| + \left| \frac{f(x)(F(\lambda)-1)}{2\lambda F(\lambda)} \right| \\ &\leqq \frac{KMxe^{-\delta v}}{|\lambda|} + \frac{KM\delta}{|\lambda|} + \left| \frac{f(x)(F(\lambda)-1)}{2\lambda F(\lambda)} \right| + \frac{Me^{-2vx}}{|\lambda|} \\ &\quad + \frac{KM}{|\lambda|^2} \int_0^\infty |P(\xi)\, d\xi. \end{aligned} \right.$$

For I_2 we have inverting the order of integration

$$(5.16)\qquad I_2 = \frac{z_1(x,\lambda)}{F(\lambda)} \int_0^x y(s,\lambda) P(s) D(x,s,\lambda)\, ds$$

where

$$D = \int_s^x f(\xi) \sin \lambda (\xi - s)\, d\xi.$$

Integrating by parts we find

$$D = -\frac{\cos \lambda (x-s)}{\lambda} f(x) + \frac{f(s)}{\lambda} + \frac{1}{\lambda} \int_s^x \cos \lambda (\xi - s) f'(\xi)\, d\xi.$$

Thus for large $|\lambda|$

$$|D| \leqq \frac{4 M e^{v(x-s)} (x+1)}{|\lambda|}.$$

Therefore

$$|I_2| \leqq \frac{MK(x+1)}{|\lambda|} \int_0^x \frac{s}{1+|\lambda| s} |P(s)|\, ds.$$

We have easily since $|\lambda| = R$ on c

$$\int_c |I_2|\, |d\lambda| \leqq MK(x+1)\pi \int_0^x \frac{s}{1+Rs} |P(s)|\, ds.$$

Thus as $R \to \infty$

$$(5.17) \qquad \int_c |I_2||d\lambda| \to 0$$

uniformly in x over any finite interval of x. From (5.15) we also have easily for $x > 0$ that as $|\lambda| = R \to \infty$

$$(5.18) \qquad \int_c \left| I_1 - \frac{f(x)}{2\lambda} \right| |d\lambda| \to 0,$$

providing we take $\delta = R^{-1/2}$, uniformly in x over any closed interval in x interior to the open interval $(0, \infty)$. But (5.17) and (5.18) complete the proof of Lemma 3.0.

In the introduction we remarked that Plancherel's theorem (1.9) holds for $f(x) \varepsilon L^2(0, \infty)$. In (3.16) we proved it for a restricted class. It is easy to exploit (3.16) to show that for any $f(x) \varepsilon L^2(0, \infty)$

$$g(u) = \underset{a \to \infty}{\text{l. i. m.}} \frac{u}{A(u)} \int_0^a f(x)\, y(x, u)\, dx$$

must exist and that

$$\int_0^\infty (g(u))^2 du = \int_0^\infty (f(x))^2 dx.$$

In case (1.0) has discrete characteristic values it is still the case that $\Phi(u)$ determines $F(\lambda)$. Indeed it can be shown that the zeros of $F(\lambda)$ which as we have seen occur at characteristic values $\lambda_k = iv_k$ are all simple. If the characteristic values are known then clearly

$$G(\lambda) = F(\lambda) \prod_{k=1}^{m} \left(\frac{1 + \dfrac{\lambda}{iv_k}}{1 - \dfrac{\lambda}{iv_k}} \right)$$

is free of zeros for $v > 0$ and thus $\log G(\lambda)$ is analytic. Moreover $|G(u)| = |F(u)|$ and

$$\arg G(u) = \arg F(u) + \sum_{k=1}^{m} \arg \left(\frac{1 + \dfrac{u}{iv_k}}{1 - \dfrac{u}{iv_k}} \right)$$

Thus $G(\lambda)$ can be found and therefore also $F(\lambda)$.

Added March 9, 1949.

The method used in proving Theorem I carries over to the equation

$$y'' + \left(u^2 - \frac{l(l+1)}{x^2} - P(x)\right) y = 0 \tag{1}$$

where l is a positive integer and $P(x)$ satisfies (1.4). Indeed if $j_l(x) = \left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(x)$ where $J_{l+\frac{1}{2}}(x)$ is the Bessel function then (1) has a solution $y(x, u)$ which satisfies

$$\lim_{x \to +0} \frac{y(x, u)}{j_l(x)} = 1. \tag{2}$$

(We recall that except for a constant $j_l(x)$ acts like x^{l+1} as $x \to 0$.) Moreover for any $u > 0$,

$$y(x, u) - \frac{A(u)}{u^{l+1}} \sin\left(ux - \frac{1}{2} l\pi - \Phi(u)\right) \to 0 \tag{3}$$

as $x \to \infty$. *It is indeed the case that* $\Phi(u)$ *determines* $P(x)$ *uniquely if*

$$\frac{l(l+1)}{x^2} + P(x) \geq 0 \tag{4}$$

(and as already stated if (1.4) is satisfied). (The condition (4) has considerably wider possibilities in application than the special case $l = 0$.)

To indicate the modifications necessary for the case $l > 0$ we introduce

$$h_l(x) = \left(\frac{\pi x}{2}\right)^{\frac{1}{2}} H^{(1)}_{l+\frac{1}{2}}(x) = e^{i\left(x - \frac{1}{2}l\pi - \frac{1}{2}\pi\right)} \left[1 - \frac{l(l+1)}{2ix} + \cdots\right]$$

where $H^{(1)}_{l+\frac{1}{2}}(x)$ is a Hankel function. Clearly $h_l(ux)$ is a solution of (1) with $P \equiv 0$ as is $h_l(-ux)$. We also have

$$j_l(x) = \frac{1}{2}\left[h_l(x) - (-1)^l h_l(-x)\right]$$

and

$$k_l(x) = \frac{1}{2i}[h_l(x)+(-1)^l h_l(-x)]$$

from which it follows that

$$j_l(x)+ik_l(x) = h_l(x).$$

If

$$\begin{cases} g(x,\xi,\lambda) = j_l(\lambda\xi)\,k_l(\lambda\xi) - j_l(\lambda\xi)\,k_l(\lambda x) \\ = \dfrac{(-1)^l}{2i}[h_l(\lambda x)\,h_l(-\lambda\xi) - h_l(-\lambda x)\,h_l(\lambda\xi)], \end{cases}$$

then the "variation of constants" formula (2.5) becomes

$$y(x,\lambda) = \frac{j_l(\lambda x)}{\lambda^{l+1}} - \frac{1}{\lambda}\int_0^x g(x,\xi,\lambda)\,P(\xi)\,y(\xi,\lambda)\,d\xi. \tag{5}$$

It is easy to show that, with $\lambda = u+iv$, and $v \geqq 0$

$$|j_l(\lambda x)| \leqq Ke^{vx}\frac{|\lambda x|^{l+1}}{(1+|\lambda x|)^{l+1}}, \quad x \leqq 0,$$

for some constant K, and also for $x \geqq \xi \geqq 0$

$$|g(x,\xi,\lambda)| \leqq Ke^{v(x-\xi)}\frac{(1+|\lambda\xi|)^l}{|\lambda\xi|^l}\,\frac{|\lambda x|^{l+1}}{(1+|\lambda x|)^{l+1}}.$$

Using these we get from (5) the analogue of Lemma (2.0) including (3). Here we also find for $v \geqq 0$ as a generalization of (2.7)

$$F(\lambda) = 1 - i\int_0^\infty \lambda^l\,y(\xi,\lambda)\,P(\xi)\,h_l(\lambda\xi)\,d\xi$$

where $A(u) = |F(u)|$ and $\Phi(u) = \arg F(u)$. Instead of (5.9) we have

$$y_1(x,\lambda) = h_l(\lambda x) + \frac{1}{\lambda}\int_x^\infty g(x,\xi,\lambda)\,P(\xi)\,y_1(\xi,\lambda)\,d\xi.$$

These indications suffice to show the changes in going from the case of Theorem I ($l = 0$) to the general case.

Added in proof: An analogue of Theorem II for $l > 0$ also holds. Interesting examples of cases where the phase does not determine the potential (owing to the presence of discrete characteristic values, i. e. bound states) have been given by V. Bargmann (Phys. Rev. 75 (1949) p. 301).

(This paper was written while the author was a John Simon Guggenheim Memorial Fellow on leave from the Massachusetts Institute of Technology.)

Literature.

[1] Fröberg, C. E. Calculation of the potential from the asymptotic phase, Arkiv för Mat. Astr. och Fysik, Band 34 A, No. 28, Band 36 A, No. 11, 1948.

[2] Titchmarsh, E. C. Eigenfunction Expansions associated with Second Order Differential Equations, Oxford 1946.

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[2] Titchmarsh, E. C. Eigenfunction Expansions associated with Second Order Differential Equations, Oxford 1946.